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LOCALLY MOST POWERFUL SEQUENTIAL TESTS FOR STOCHASTIC PROCESSES

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For a continuous time stochastic process with distribution P_ϑ depending on a one-dimensional parameter ϑ the problem of sequentially testing $\vartheta = 0$ against $\vartheta > 0$ is treated. We assume that the process of likelihood ratios has a certain representation which allows to obtain identities of the Wald type for stopping times. These identities are then used to derive a result on locally most powerful tests for which a problem of optimal stopping is solved.

Sequential tests	locally most powerful tests
Wald identities	likelihood ratio process
optimal stopping	

1. Introduction and preliminaries

We consider a continuous time stochastic process whose distribution P_ϑ depends on a one-dimensional parameter ϑ, and treat the problem of sequentially testing $\vartheta = 0$ against $\vartheta > 0$. In the discrete time parameter case, when observing an i.i.d. sequence of random variables, this problem was treated by Abraham [1] and Berk [2] who obtained the following result: Certain tests defined similar to SPRT's are locally most powerful tests in the sense that they maximize the slope of the power function in $\vartheta = 0$ under all sequential tests with the same power and no greater expected sample size for $\vartheta = 0$. Using Berk's approach of considering a problem of optimal stopping, Irle [6] obtained similar results for continuous time Gaussian processes.

A discussion of sequential tests for continuous time stochastic processes can be found, e.g., in [4] where Dvoretzky, Kiefer and Wolfowitz considered two simple hypotheses, and [3] where Brown constructed unbiased tests for testing $\vartheta = 0$ against $\vartheta \neq 0$.

Representing the available information in the sequential model by an increasing family $(\mathcal{F}_t)_{t\in T}$ of σ-algebras, we assume in this paper that the densities $X_{\vartheta,t} = \mathrm{d}P_\vartheta|\mathcal{F}_t/\mathrm{d}P|\mathcal{F}_t$ can be represented as $X_{\vartheta,t} = \exp(\vartheta Y_t - \frac{1}{2}\vartheta^2 Z_t)$ where the stochastic processes $(Y_t)_{t\in T}$ and $(Z_t)_{t\in T}$ fulfill some additional requirements. It is shown that this representation readily yields identities of the Wald type for stopping times which

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are then used to obtain a result on locally most powerful tests. Our assumptions imply that we are in the case of conditional exponential families as defined by Feigin [5].

The situation which we consider in this paper includes the case of Gaussian processes as treated by Irle [6], and also the case of certain Ito processes as treated, e.g., by Liptser and Shiryayev [11, Section 17.6] for the problem of sequentially testing two simple hypotheses.

Let $(\Omega, \mathscr{A}, P)$ be a probability space, $T = [0, \infty)$ and $(\mathscr{F}_t)_{t \in T}$ a right-continuous increasing family of sub-σ-algebras of $\mathscr{A}$. Let U be an open interval containing 0 and for every $\vartheta \in U$ let P_ϑ be a probability measure on $(\Omega, \mathscr{A})$ such that for every $t \in T$, $P_\vartheta|\mathscr{F}_t$ is dominated by $P|\mathscr{F}_t$. Here $P_\vartheta|\mathscr{F}_t$ $(P|\mathscr{F}_t)$ denotes the restriction of P_ϑ (P) to $\mathscr{F}_t$. We consider versions $X_{\vartheta,t}$ of the densities $\mathrm{d}P_\vartheta|\mathscr{F}_t/\mathrm{d}P|\mathscr{F}_t$ such that for every $\vartheta \in U$ the stochastic process $(X_{\vartheta,t})_{t\in T}$ is a local martingale, thus right-continuous (see, e.g. [7, Chapter 7.1]).

A mapping $\tau: \Omega \to [0, \infty]$ is called a stopping rule iff for every $t \in [0, \infty)$ the set $\{\tau \leq t\}$ belongs to $\mathscr{F}_t$, and a stopping rule is called a stopping time iff $\mathbf{P}(\{\tau = \infty\}) = 0$. Furthermore we set $\mathscr{F}_\tau = \{A \in \mathscr{A} : A \cap \{\tau \leq t\} \in \mathscr{F}_t \text{ for all } t\}$. It follows that for every stopping rule

$$X_{\vartheta,\tau}|\{\tau < \infty\} = \frac{\mathrm{d}P_\vartheta|\mathscr{F}_\tau \cap \{\tau < \infty\}}{\mathrm{d}P|\mathscr{F}_\tau \cap \{\tau < \infty\}}$$

(see, e.g. [7, Chapter 7.1]).

We now make the following assumptions:

(A1) There exist right-continuous stochastic processes $(Y_t)_{t\in T}$ and $(Z_t)_{t\in T}$ adapted to $(\mathscr{F}_t)_{t\in T}$ such that for every $t \in T$

$$X_{\vartheta,t} = \exp(\vartheta Y_t - \tfrac{1}{2}\vartheta^2 Z_t) \quad \text{for every } \vartheta \in U.$$

(A2) $(Z_t)_{t\in T}$ has increasing paths ≥ 0; there exists an increasing sequence of stopping rules τ_k $(k \in \mathbb{N})$ with $\sup \tau_k = \infty$ such that for every k

$$\mathbf{E}\exp(\gamma_k Z_{\tau_k}) < \infty \quad \text{for some } \gamma_k > 0$$

where $Z_\infty = \sup Z_t$.

Lemma 1.1. *For every stopping time τ with $\mathbf{E}Z_\tau < \infty$, $\mathbf{E}Y_\tau = 0$ and $\mathbf{E}Y_\tau^2 = \mathbf{E}Z_\tau$ holds.*

Proof. It follows from the assumptions that $(Y_t)_{t\in T}$ and $(Y_t^2 - Z_t)_{t\in T}$ are local martingales such that the identities

$$\mathbf{E}Y_{\tau\wedge\tau_k} = 0 \quad \text{and} \quad \mathbf{E}(Y^2_{\tau\wedge\tau_k} - Z_{\tau\wedge\tau_k}) = 0$$

are valid for any stopping time τ (see [13, p. 499] where for continuous $(Y_t)_{t\in T}$ the process $(Z_t)_{t\in T}$ is identified as $\langle (Y_t)_{t\in T}, (Y_t)_{t\in T}\rangle$, the associated increasing process). Consider now a stopping time τ with $\mathbf{E}Z_\tau < \infty$. It follows that

$$\mathbf{E}Y_\tau^2 \leq \liminf \mathbf{E}Y^2_{\tau\wedge\tau_k} = \mathbf{E}Z_\tau$$

which implies that

$$\liminf \int_{\{\tau>\tau_k\}} |Y_{\tau_k}| \, \mathrm{d}P = 0,$$

and thus $\mathbf{E} Y_\tau = 0$. From this we obtain

$$\mathbf{E}(Y_\tau | \mathscr{F}_{\tau\wedge\tau_k}) = Y_{\tau\wedge\tau_k}$$

and so

$$\mathbf{E}(Y_\tau^2 | \mathscr{F}_{\tau\wedge\tau_k}) I_{\{\tau>\tau_k\}} \geqslant Y^2 \tau_k I_{\{\tau>\tau_k\}}$$

which implies

$$\mathbf{E} Y_\tau^2 \geqslant \mathbf{E} Y_{\tau\wedge\tau_k}^2 \quad \text{for all } k,$$

and thus $\mathbf{E} Y_\tau^2 \geqslant \mathbf{E} Z_\tau$.

2. Locally best tests

Let us now consider the problem of testing H: $\vartheta = 0$ against K: $\vartheta > 0$. A sequential test consists of a stopping rule τ and a mapping $\psi : \Omega \to [0, 1]$ such that ψ is $\mathscr{F}_\tau$-measurable and $\psi \cdot I_{\{\tau=\infty\}} = 0$.

For a sequential test its power function $\beta(\tau, \psi)$ is defined as $\beta(\tau, \psi)(\vartheta) = \mathbf{E}_\vartheta \psi = \mathbf{E}_\vartheta \psi I_{\{\tau<\infty\}}$ for $\vartheta \in U$, thus

$$\beta(\tau, \psi)(\vartheta) = \mathbf{E}\psi \exp(\vartheta Y_\tau - \tfrac{1}{2}\vartheta^2 Z_\tau).$$

As a measure for the slope of the power function in 0 we consider for any sequential test the quantity

$$\lambda(\tau, \psi) = \limsup_k \mathbf{E}\psi Y_\tau I_{\{\tau<\tau_k\}}$$

which is motivated by the following considerations.

By formal differentiation under the expectation sign of the power function we obtain for $\vartheta = 0$ the expression $\mathbf{E}\psi Y_\tau$, and so, if this formal operation is justified and $\mathbf{E}\psi Y_\tau$ exists, the derivative of the power function for $\vartheta = 0$ is equal to $\lambda(\tau, \psi)$. Furthermore for $\psi_k = \psi I_{\{\tau<\tau_k\}}$ it follows from the theory of exponential families that

$$\frac{\mathrm{d}}{\mathrm{d}\vartheta} \beta(\tau, \psi_k)(0) = \mathbf{E}\psi_k Y_\tau = \mathbf{E}\psi Y_\tau I_{\{\tau<\tau_k\}},$$

thus $\lambda(\tau, \psi)$ defines an extended real number for every sequential test.

In the following we will find tests which maximize $\lambda(\tau, \psi)$ under certain restrictions.

For real numbers a, b with $a<b$ define a stopping rule

$$\sigma(a, b)=\inf\{t\colon Y_t \notin (a, b)\},$$

and let $\psi(a, b)=I_{\{Y_{\sigma(a,b)}\geq b\}}$.

Additionally to (A1) and (A2) we make the following assumptions:

(A3) $(Y_t)_{t\in T}$ has continuous paths, and $Y_0=0$.

(A4) $\sup Z_t=\infty$.

Under (A3), $(Y_t)_{t\in T}$ is a time-transformed Wiener-process (see [9]), and one could approach the problem in this paper by using this fact and the reasoning of Brown [3]. In the following we take a simple direct approach not using the result of Kunita and Watanabe.

Theorem 2.1. *For any real numbers a, b with $a<b$ the sequential test $(\sigma(a, b), \psi(a, b))$ has the following properties.*

(i) $\mathbf{P}(\{\sigma(a, b)<\infty\})=1$ *and* $\mathbf{E}Z_{\sigma(a,b)}<\infty$.

(ii) *For any sequential test (τ, ψ) with $\mathbf{E}Z_\tau \leq \mathbf{E}Z_{\sigma(a,b)}$ and $\beta(\tau, \psi)(0)=\beta(\sigma(a, b), \psi(a, b))$ the following inequality holds*:

$$\lambda(\tau, \psi)\leq \lambda(\sigma(a, b), \psi(a, b)).$$

Proof. (i) From (A3) it follows for every k that

$$\max\{a^2, b^2\}\geq \mathbf{E}Y^2_{\sigma(a,b)\wedge\tau_k}=\mathbf{E}Z_{\sigma(a,b)\wedge\tau_k},$$

thus by monotone convergence

$$\mathbf{E}Z_{\sigma(a,b)}\leq \max\{a^2, b^2\}<\infty$$

and $\mathbf{P}(\{\sigma(a, b)<\infty\})=1$ by (A4).

(ii) The following argument is similar to the argument given in Irle [6] so that we omit some of the details, but it differs in the aspect that here—as remarked above—we do not use a time-transformation argument.

(1) Choose real numbers c, d with $c>0$, $a=d-1/4c$ and $b=d+1/4c$.

The following inequality is easily proven:

$$\lambda(\tau, \psi)-d\beta(\tau, \psi)(0)\leq \mathbf{E}(Y_\tau-d)^+I_{\{\tau<\infty\}}.$$

(2) Consider now the problem of optimal stopping for

$$R_t=(Y_t-d)^+I_T(t)-cZ_t.$$

We will prove that $\{R^+_\tau\colon \tau \text{ stopping rule}\}$ is uniformly integrable.

Since for any $0<c'<c$,

$$R^+_t\leq (Y^+_t-c'Z_t)^++d^-$$

we may assume $d=0$ and $2c\in U$. For any stopping rule τ we have

$$\exp((2cY_\tau-2c^2Z_\tau)^+)I_{\{\tau<\infty\}}\leq 1+\exp(2cY_\tau-2c^2Z_\tau)I_{\{\tau<\infty\}}$$

and thus

$$\mathbf{E}\exp(2c(Y_\tau - cZ_\tau)^+)I_{\{\tau<\infty\}} \leqslant 2.$$

This implies by LaVallée–Poussin's theorem that

$$\{(Y_\tau - cZ_\tau)^+ I_{\{\tau<\infty\}}: \tau \text{ stopping rule}\}$$

is uniformly integrable, and the first part of the assertion follows from the equality

$$R_\tau^+ = (Y_\tau - cZ_\tau)^+ I_{\{\tau<\infty\}}.$$

This yields furthermore $\sup \mathbf{E}R_\tau^+ < \infty$, and from $-cZ_\tau \leqslant R_\tau \leqslant ((Y_\tau - d)^+ - \frac{1}{2}cZ_\tau)^+ - \frac{1}{2}cZ_\tau$ we obtain $\mathbf{E}R_\tau > -\infty$ iff $\mathbf{E}Z_\tau < \infty$, and thus

$$v = \sup_\tau \mathbf{E}R_\tau = \sup_{\tau, \mathbf{E}Z_\tau < \infty} \mathbf{E}R_\tau < \infty.$$

(3) We will now compute $\mathbf{E}R_{\sigma(a,b)}$.

If $0 \notin [a, b]$, then $\sigma(a, b) = 0$ and $\mathbf{E}R_{\sigma(a,b)} = d^-$.

Assume now $0 \in [a, b]$. From (i):

$$\mathbf{P}(\{Y_{\sigma(a,b)} = a\}) = 1 - \mathbf{P}(\{Y_{\sigma(a,b)} = b\})$$

and from Lemma 1.1:

$$a\mathbf{P}(\{Y_{\sigma(a,b)} = a\}) + b\mathbf{P}(\{Y_{\sigma(a,b)} = b\}) = 0.$$

This yields

$$\mathbf{P}(\{Y_{\sigma(a,b)} = a\}) = \frac{b}{b-a} \quad \text{and} \quad \mathbf{P}(\{Y_{\sigma(a,b)} = b\}) = \frac{a}{a-b},$$

from which we obtain by using Lemma 1.1 again and (A3)

$$\mathbf{E}R_{\sigma(a,b)} = (b-d)\frac{a}{a-b} - c(-ab) = cd^2 - \frac{d}{2} + \frac{1}{16c}.$$

(4) To show that $\sigma(a, b)$ is an optimal stopping time we will prove $v \leqslant \mathbf{E}R_{\sigma(a,b)}$.

Define a real function h by

$$h(y) = cy^2 + (\tfrac{1}{2} - 2cd)y + \left(cd^2 - \frac{d}{2} + \frac{1}{16c}\right).$$

Then $h(y) \geqslant (y-d)^+$ for all y. It follows from Lemma 1.1 that $\mathbf{E}h(Y_\tau) = c\mathbf{E}Z_\tau + h(0)$ for all τ with $\mathbf{E}Z_\tau < \infty$, and thus from (2),

$$\begin{aligned} v &= \sup_{\tau, \mathbf{E}Z_\tau<\infty} \mathbf{E}R_\tau \\ &\leqslant \sup_{\tau, \mathbf{E}Z_\tau<\infty} \mathbf{E}(h(Y_\tau) - cZ_\tau) = h(0) = cd^2 - \frac{d}{2} + \frac{1}{16c}. \end{aligned}$$

This implies the assertion for $0 \in [a, b]$.

Assume $0 \notin [a, b]$ and define a real function g by

$$g(y) = \begin{cases} 0 & \text{if } y \in [a, b], \\ (y-d)^+ - h(y) & \text{if } y \notin [a, b]. \end{cases}$$

Then $g(y) \geqslant (y-d)^+ - h(y)$ for all y, and g is concave. This implies for τ with $\mathbf{E}Z_\tau < \infty$ that

$$\mathbf{E}((Y_\tau - d)^+ - h(Y_\tau)) \leqslant \mathbf{E}g(Y_\tau) \leqslant g(0)$$

and thus

$$v = \sup_{\tau, \mathbf{E}Z_\tau < \infty} \mathbf{E}((Y_\tau - d)^+ - h(Y_\tau)) + h(0) \leqslant g(0) + h(0) = d^-.$$

(5) Now for any sequential test (τ, ψ) with $\mathbf{E}Z_\tau < \infty$ we have from (1) that

$$\begin{aligned} \lambda(\tau, \psi) - d\beta(\tau, \psi)(0) - c\mathbf{E}Z_\tau &\leqslant \mathbf{E}((Y_\tau - d)^+ I_{\{\tau<\infty\}} - cZ_\tau) \\ &\leqslant \mathbf{E}R_{\sigma(a,b)} = \lambda(\sigma(a, b), \psi(a, b)) - d\beta(\sigma(a, b), \psi(a, b)) - c\mathbf{E}Z_{\sigma(a,b)}. \end{aligned}$$

From this the assertion (ii) follows at once.

Let us remark that the second part of assumption (A3), i.e. $Y_0 = 0$, could be removed by conditioning as in [6, (1.5)]. There it is also shown that part (ii) of the theorem is no longer valid in general for sequential tests (τ, ψ) satisfying $\mathbf{E}\tau \leqslant \mathbf{E}\sigma(a, b)$ instead of $\mathbf{E}Z_\tau \leqslant \mathbf{E}Z_{\sigma(a,b)}$.

3. Applications

We consider a stochastic process in its function space representation, i.e. let Ω be $\mathbb{R}^T$, and denote by V_t the coordinate mapping of index t. Let $\mathcal{A}$ be the σ-algebra induced by $(V_t)_{t\in T}$, and $(\mathcal{F}_t)_{t\in T}$ the usual right-continuous family induced by observing the coordinate process.

(i) Let P be Wiener measure on $(\Omega, \mathcal{A})$, and assume that there is given a signal process $(S_t)_{t\in T}$ on $(\Omega, \mathcal{A})$. For $\vartheta \in \mathbb{R}$ let P_ϑ be the probability measure on $(\Omega, \mathcal{A})$ which is induced by the Ito process with differential $\vartheta S_t\,\mathrm{d}t + \mathrm{d}W_t$ where W_t is a Wiener process on $(\Omega, \mathcal{A})$. We thus have $P = P_0$, and the problem of testing $\vartheta = 0$ against $\vartheta > 0$ amounts to a signal detection problem.

In this situation conditions which yield the equivalence of the measures $P_\vartheta \mid \mathcal{F}_t$ and $P|\mathcal{F}_t$ ($\vartheta \in \mathbb{R}$, $t \in T$), and formulas for the densities are well known (see, e.g. [8] and [10, Chapter 7]). Omitting the exact conditions for the validity we obtain from Kailath [8] (assuming that each $\mathcal{F}_t$ is augmented by all P-zero sets) for all t

$$\frac{\mathrm{d}P_\vartheta|\mathcal{F}_t}{\mathrm{d}P|\mathcal{F}_t} = \exp\left(\vartheta \int_0^t \hat{S}_u\,\mathrm{d}V_u - \frac{\vartheta^2}{2} \int_0^t \hat{S}_u^2\,\mathrm{d}u\right)$$

with $\hat{S}_u = \mathbf{E}(S_u|\mathscr{F}_u)$ wh... ve may assume that the stochastic processes

$$(Y_t)_{t\in T} = \left(\int_0^t \hat{S}_u \, dV_u\right)_{t\in T} \quad \text{and} \quad (Z_t)_{t\in T} = \left(\int_0^t \hat{S}_u^2 \, du\right)_{t\in T}$$

have continuous paths.

So assumptions (A1), (A3) and the first part of (A2) are easily seen to hold. Furthermore the second part of (A2) is also fulfilled, and we may choose $\tau_k = \inf\{t: Z_t = k\}$, so that only (A4) has to be required separately.

(ii) For a given covariance function $R \in \mathbb{R}^{T\times T}$ and a given $m \in \mathbb{R}^T$ $(m \neq 0)$ let $(P_\vartheta)_{\vartheta\in\mathbb{R}}$ denote the family of Gaussian measures on $(\Omega, \mathscr{A})$ with common covariance kernel R and mean value function ϑm, and let $P = P_0$. Conditions for the equivalence of $P_\vartheta|\mathscr{F}_t$ and $P|\mathscr{F}_t$, and formulas for the densities are well known (see, e.g. [12]) and one obtains

$$\frac{dP_\vartheta|\mathscr{F}_t}{dP|\mathscr{F}_t} = \exp(\vartheta Y_t - \tfrac{1}{2}\vartheta^2 Z_t)$$

where $(Z_t)_{t\in T}$ is non-random.

A detailed treatment of this situation is given in [6]. Let us just remark here that in this situation we may choose $\tau_k = k$ in (A2), since Z_k is a (finite) constant.

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